Control Systems : Set 10 : Statespace (1) - Solutions

Prob 1 | Given the system

$$\dot{x} = \begin{bmatrix} -5 & 1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

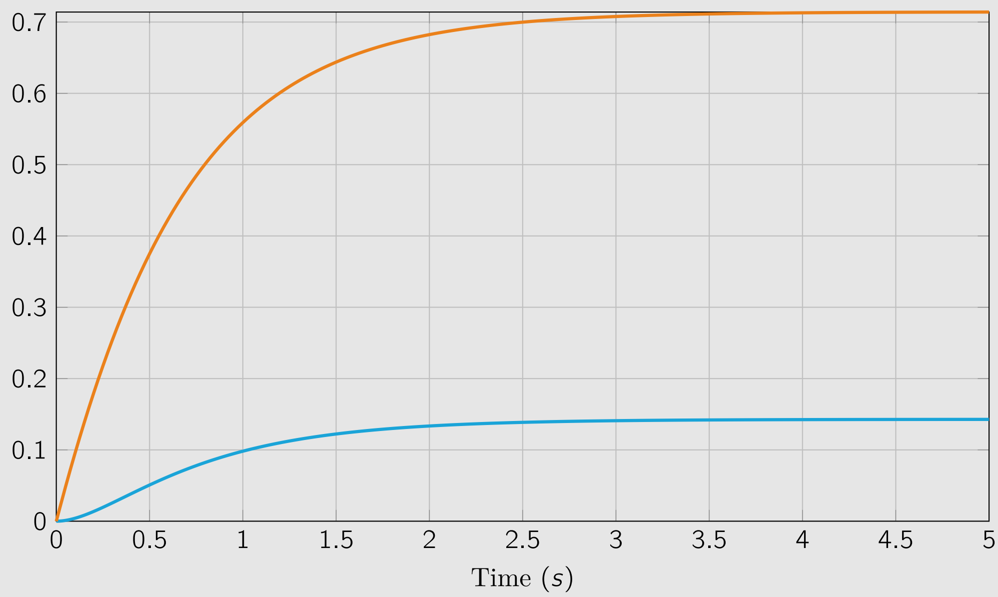
with zero initial conditions, find the steady-state value of x for a step input u.

We are given $\dot{x} = Ax + Bu$. Steady-state means that $\dot{x} = 0$ and a step input (or unit step) means $u = 1(t)$. Thus, assuming that the system is stable and A is invertible (which you can check), we have

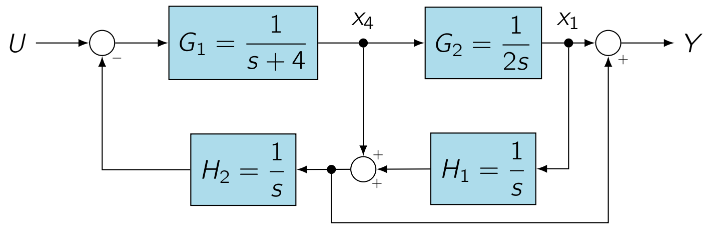
$$\begin{aligned} 0 &= Ax_{ss} + B \cdot 1 \\ \Rightarrow x_{ss} &= -A^{-1}B \\ &= \begin{bmatrix} -5 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ \frac{5}{7} \end{bmatrix} \end{aligned}$$

This can be verified in Matlab with the following code, as can be seen in the plot below.

```
A = [-5 1; -2 -1];
B = [0;1];
C = eye(size(A));
D = [0;0];
sys = ss(A,B,C,D);
step(sys,5)
dcgain(sys)
```



Prob 2 | Consider the system shown below



a) Find the transfer function from U to Y

$$\left.\begin{aligned} Y &= x_1 + x_4 + \frac{1}{s}x_1 \\ x_1 &= \frac{1}{2s}x_4 \end{aligned}\right\} \quad \rightarrow \quad Y = \frac{2s^2+s+1}{2s^2}x_4$$

$$x_4 = \frac{1}{s+4}\left(U - \frac{1}{s}\left(x_4 + \frac{1}{2s^2}x_4\right)\right) \quad \rightarrow \quad \frac{x_4}{U} = \frac{2s^3}{2s^4+8s^3+2s^2+1}$$

Putting these together gives

$$\frac{Y}{U} = \frac{s(2s^2+s+1)}{2s^4+8s^3+2s^2+1}$$

b) Write state equations for the system using the state variables indicated

Write out each of the states given in the time domain

$$\begin{aligned} x_1 &= \frac{1}{2s}x_4 & &\rightarrow & \dot{x}_1 &= \frac{1}{2}x_4 \\ x_2 &= \frac{1}{s}x_1 & &\rightarrow & \dot{x}_2 &= x_1 \\ x_3 &= \frac{1}{s}(x_4 + x_2) & &\rightarrow & \dot{x}_3 &= x_4 + x_2 \\ x_4 &= \frac{1}{s+4}(u - x_3) & &\rightarrow & \dot{x}_4 &= -4x_4 + u - x_3 \\ y &= x_1 + x_2 + x_4 & & & & \end{aligned}$$

Writing in standard state-space form gives

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$$

Prob 3 | Show that a transfer function is not changed by a linear transformation of the state.

Assume the original system is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ G(s) &= C(sI - A)^{-1}B + D \end{aligned}$$

Assume a change of state from x to z using the nonsingular transformation T

$$x = Tz$$

The new system matrices are

$$\bar{A} = T^{-1}AT \qquad \bar{B} = T^{-1}B \qquad \bar{C} = CT \qquad \bar{D} = D$$

The transfer function is

$$\begin{aligned} G_z(s) &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \end{aligned}$$

If we factor T on the left and T^{-1} on the right of the $(sI - T^{-1}AT)^{-1}$ term, we obtain

$$\begin{aligned} G_z(s) &= CT(sTT^{-1} - T^{-1}AT)^{-1}T^{-1}B + D \\ &= CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= C(sI - A)^{-1}B + D \\ &= G(s) \end{aligned}$$

Prob 4 | For each of the listed transfer functions, write the state equations in control canonical form. In each case draw a block diagram and give the appropriate expressions for A, B and C.

a) $G(s) = \dfrac{s^2 - 2}{s^2(s^2 - 1)}$

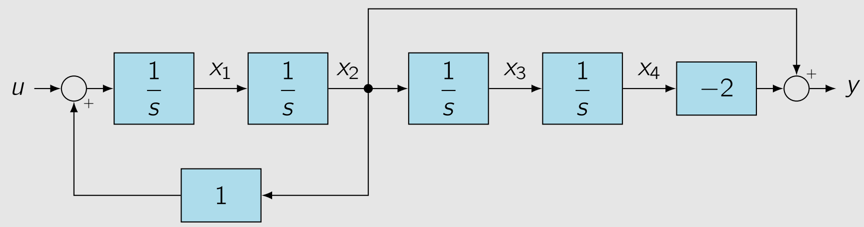
Re-write in standard form and then read off the coefficients

$$G(s) = \frac{s^2 - 2}{s^4 - s^2}$$

Control canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

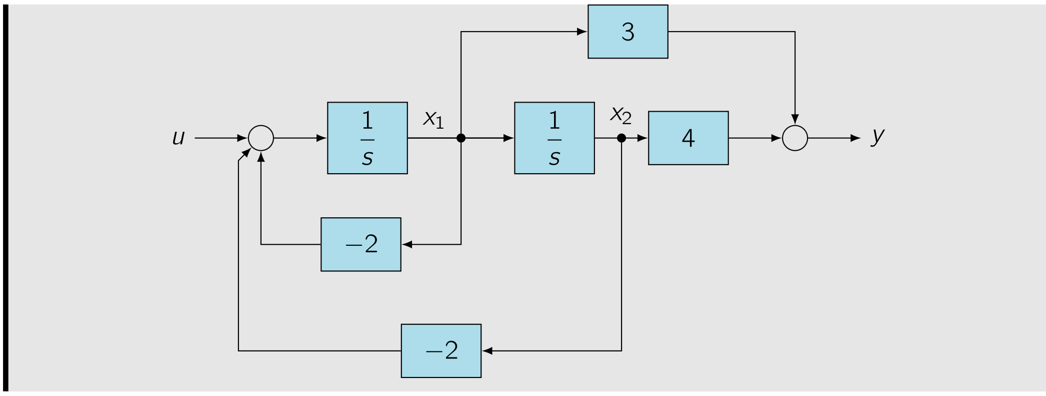


b) $G(s) = \dfrac{3s + 4}{s^2 + 2s + 2}$

This one can be directly translated into control canonical form

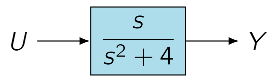
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



u
1/s
x1
1/s
x2
4
y
3
−2
−2

Prob 5 | Consider the system in the figure below.



a) Write a set of equations that describes this system in the control canonical form as $\dot{x} = Ax + Bu$ and $y = Cx$

> There are two poles, so we will have two states
>
> $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
>
> $$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Design a control law of the form

$$u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which will place the closed-loop poles at $s = -2 \pm 2j$

> The desired characteristic equation is
>
> $$(s + 2 + 2j)(s + 2 - 2j) = s^2 + 4s + 8$$
>
> Since the system is in control canonical form, we can immediately write down the control law as
>
> $$K = \begin{bmatrix} 0 + 4 & -4 + 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \end{bmatrix}$$

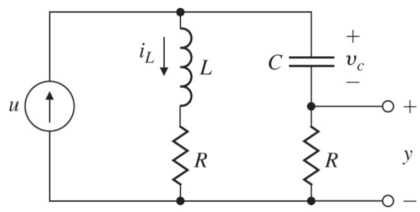
c) Check your answer with Matlab

```
A = [0  -4;  1  0];
B = [1;0];
p = [-2+2*j,  -2-2*j];
K = place(A,B,p)

K =

        4              4
```

Prob 6 | Consider the electric circuit shown in the figure below



a) Write the internal (state) equations for the circuit. The input $u(t)$ is a current, and the output y is a voltage. Let $x_1 = i_L$ and $x_2 = v_c$

Recall the the voltage drop across an inductor is $v = L\dot{i}$, and the current through a capacitor is $i = C\dot{v}$.

We can write the dynamic equations for this circuit as

$$\begin{aligned} \dot{i}_L L + R i_L &= v_c + RC\dot{v}_c && \text{Voltage drop across the two paths is the same} \\ C\dot{v}_c + i_L &= u && \text{Current into the top node is equal to the current out} \\ y &= (u - i_L)R && \text{Ohm's law gives the voltage drop across the output resistor} \end{aligned}$$

Re-write in matrix form

$$\begin{bmatrix} L & -RC \\ 0 & C \end{bmatrix} \begin{pmatrix} \dot{i}_L \\ \dot{v}_c \end{pmatrix} = \begin{bmatrix} -R & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and solve for the derivative of the state

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_c \end{pmatrix} = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{pmatrix} \frac{R}{L} \\ \frac{1}{C} \end{pmatrix} u$$

$$y = \begin{bmatrix} -R & 0 \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + Ru$$

b) What condition(s) on R, L and C will guarantee that the system is controllable?

The system is controllable if and only if the controllability matrix is full rank.

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & \frac{L-CR^2}{CL^2} \\ \frac{1}{C} & -\frac{R}{CL} \end{bmatrix}$$

Test for uncontrollability by checking when the determinant is zero

$$\begin{aligned} \det(\mathcal{C}) &= 0 \\ &= -\frac{1}{C^2 L} \end{aligned}$$

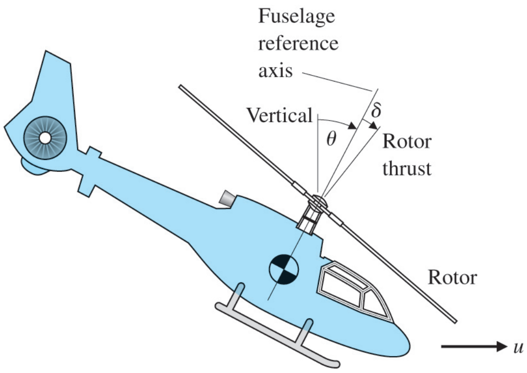
Therefore the system will be controllable for all (finite) capacitors and inductors.

Prob 7 | The linearized longitudinal motion of a helicopter near hover (figure below) can be modeled by the normalized third-order system

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ u \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta$$

where

$q =$ pitch rate
$\theta =$ pitch angle of fuselage
$u =$ horizontal velocity (standard aircraft notation)
$\delta =$ rotor tilt angle (control variable)



Suppose our sensor measures the horizontal velocity u as the output, that is $y = u$
Use Matlab to answer the following questions.

a) Find the open-loop pole locations.

> Poles are the eigenvalues of A: -0.66, $0.12 \pm 0.37i$, These can be computed either with eig(A) or with the command pole.

b) Is the system controllable?

> Check the rank of the controllability matrix
>
> $$\text{rank}(\mathcal{C}) \begin{bmatrix} B & AB & A^2B \end{bmatrix} = 3$$
>
> If we check the singular values with Matlab, we get 67.1, 7.1 and 5.2. We can conclude that the matrix is nowhere near singular, and so this system is not close to being uncontrollable.

c) Find the feedback gain that places the poles of the system at $s = -1 \pm 1j$ and $s = -2$

Using the Matlab place command, we get the controller

$$K = \begin{bmatrix} 0.47 & 1 & 0.063 \end{bmatrix}$$